On the Motion of a Perfect Incompressible Fluid when no Solid Bodies are Present.

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CONTENTS.

In a paper published in this Journal recently, the analogy of an electric current in fluid motion was found to be what I there called the *vortices* of the *vortices* of the fluid. It soon became evident to me that the kind of motion there referred to was a motion which was higher than vortex motion and yet had some of its properties.

The first portion of the present paper, treating of vector quantities in general, shows how to deduce the higher vectors from the lower, and also gives a method by which from three integrals any lower system can be deduced from any higher system. One of the most useful theorems, both for use in hydrodynamics and electro-magnetism, is that which shows how to replace any system of vectors by another system having the same external effect or the same internal effect, as the case may be. Another useful theorem shows how to deduce a system of vectors satisfying the equation of continuity from one which does not, though the principle of this theorem has long been recognized in the idea of cyclic constants. But I believe it will be found more useful as stated in this paper.

Application is then made to the kinematics of fluid motion.

Fluid motion must satisfy the equation of continuity, and so must be always cyclic. Otherwise there must be points somewhere in space where the fluid is created or destroyed.

As Laplace's equation is a form of the equation of continuity, it is evident that this equation cannot, *from a physical point of view*, be satisfied at every point

of space, although there are some functions, like the spherical harmonics of positive degree, that do so mathematically; for these all give some fluid motion, often infinite in amount, at an infinite distance. So that such fluid motion can only be produced by the creation and destruction of fluid in regions infinitely distant. And so all fluid motion in an indefinite medium, without solid bodies, must be due, as it were, to the presence of vortex filaments or filaments of any of the higher motions.

The n^{th} order of motion always satisfies the equation of continuity, and hence the lines of motion are closed. All the infinite series of motions must exist *somewhere* in space, but they may be confined to a point, a line, or a surface, and so, for all other points of space, only the n^{th} and lower orders may exist.

In this case, where the $(n+1)^{\text{th}}$ order is zero throughout a certain region, the n^{th} order will have a potential satisfying Laplace's equation throughout that region, though it may be cyclic. The motions of a fluid can be divided into two varieties, rotatory and translatory, the odd orders being rotatory and the even ones translatory.

The motion of the zero order is ordinary irrotational fluid motion; the first order is vortex motion; the second order is what I have called the *vortices* of the *vortices,* or *relative fluid motion,*— in an element possessing this motion the centre moves faster than the sides; the third order is one vortex filament inside another and rotating in the opposite direction: and so we can proceed. We thus get the idea that all the higher motions are, as it were, motions which take place inside the element as it drifts along.

From this we see that when we deal with the higher motions, especially with any discontinuity in their distribution, we must use a smaller element than we otherwise should have to do.

Thus, when we treat a single vortex filament, it is impossible to get the motion of the filament or the energy of the fluid without an integration across the section of the filament, however small this may be. This principle I consider of the utmost importance, and by recognition of it all that has hitherto puzzled us with respect to the motion of a vortex ring or the stability of fluid motion becomes a mere question of ordinary calculation. For, if we attempt to obtain the motion of translation of a circular vortex filament, we only obtain indeterminate results if we treat the filament as a whole. For I here show that the motion of the fluid exterior to the filament is the same for an infinite number of ways in which the vortex motion is distributed within the filament, and each of these ways has its own motion of translation and its own energy.

A method is here given by which any distribution of motion of the n^{th} order can be replaced by another distribution over a surface, so that the action on the

other side of the surface away from the original distribution shall be the same as before. This leads us, in the case of vortex motion, to the formation of hollow vortex rings, or solid ones of any finite cross section, and it is shown that one form of hollow vortex ring is permanent and probably stable.

For I do not admit the validity of Thompson's proof that *all* discontinuous fluid motion is unstable. Illustrations are given of the distributions of motions of the different orders, and it is shown that any such series can represent a number of cases of fluid motion, seeing that we can take any one of the series of motions as the ordinary or irrotational motion.

In treating of the action of forces on a fluid, we see that a single attracting particle can only produce fluid pressure, but no motion. Hence in an unlimited fluid forces having an acyclic potential can have no effect on the fluid motion, but can only produce a distribution of pressure.

When a solid body is immersed in a fluid, the fluid is no longer continuous, but the surface acts like a diaphragm; in this case we can have an acyclic velocity potential outside the body, but cyclic if the body was removed. But where the fluid is unlimited this cannot be so.

The motion of a solid body in a fluid can always be represented by the proper case of motion in an unlimited medium, the condition being that the components of the fluid motion throughout a certain surface representing the surface of the body shall be the same as the components of the motion of an element of the surface of the body at that point, supposing the body to exist.

As the classification of vectors is general, I here apply the method to forces, and so get the idea of forces of different orders.

All forces which can produce fluid motion in an unlimited medium must be cyclic, and must contain all the infinite series of forces somewhere in space, though any given region may contain only forces of the n^{th} and lower orders, the n^{th} having a potential.

Expressions are then given for the fluid pressure in terms of the motion of the fluid, one of which has already been given by Dr. Craig.

I have not yet given any expressions for the energy of the fluid in terms of the higher orders of motion, though that in terms of vortex motion is well known.

An immense number of applications of the ideas here set forth of course suggest themselves, but most of the results so far obtained are set forth in this paper, as I have not yet had time to descend to details. It is of course possible that some of the conclusions will not stand the test of time, but I hope that enough will remain to make the paper of value in all departments of physics where vector quantities are investigated.

The different portions of the paper have not been written in the order here given, and so the same idea is sometimes repeated in different parts.

Neither is the notation very satisfactory, as different letters are sometimes used in the same sense, though in this case the connection is stated. I have changed the notation so often that I despair of perfection, and so publish the paper as it stands.

General Theory of Vector Quantities.

Let there be *any* distribution of a vector quantity, $\bar{M}_0$, throughout space, and let its components be $\bar{F}_0$, $\bar{G}_0$, and $\bar{H}_0$. The dash over the letter signifies that these quantities do not satisfy the equation of continuity. Let us now subject these quantities to the following operations:—

$$F_1 = \frac{d\bar{H}_0}{dy} - \frac{d\bar{G}_0}{dz},$$

$$G_1 = \frac{d\bar{F}_0}{dz} - \frac{d\bar{H}_0}{dx},$$

$$H_1 = \frac{d\bar{G}_0}{dx} - \frac{d\bar{F}_0}{dy},$$

$$F_2 = \frac{dH_1}{dy} - \frac{dG_1}{dz},$$

$$G_2 = \frac{dF_1}{dz} - \frac{dH_1}{dx},$$

$$H_2 = \frac{dG_1}{dx} - \frac{dF_1}{dy},$$

$$F_3 = \frac{dH_2}{dy} - \frac{dG_2}{dz},$$

etc.

And let us continue substitutions of this nature indefinitely.

We have thus derived from $\bar{F}_0$, $\bar{G}_0$, and $\bar{H}_0$ an infinite number of new systems of vectors whose properties and relations to the original vectors are to be studied. In the first place, we notice that all these new quantities satisfy the equation of continuity except $\bar{F}_0$, $\bar{G}_0$, and $\bar{H}_0$, and hence we have

$$\frac{dF_n}{dx} + \frac{dG_n}{dy} + \frac{dH_n}{dz} = 0.$$

From these equations we may find, writing J_0 for the quantity

$$\frac{d\bar{F}_0}{dx} + \frac{d\bar{G}_0}{dy} + \frac{d\bar{H}_0}{dz},$$

the following relations: —

$$F_2 = \frac{dJ_0}{dx} - \Delta^2 \overline{F}_0,$$

$$G_2 = \frac{dJ_0}{dy} - \Delta^2 \overline{G}_0,$$

$$H_2 = \frac{dJ_0}{dz} - \Delta^2 \overline{H}_0,$$

and for the rest,

$$F_n = -\Delta^2 F_{n-2},$$

$$G_n = -\Delta^2 G_{n-2},$$

$$H_n = -\Delta^2 H_{n-2}.$$

Whence, writing χ_0 for the quantity

$$\chi_0 = \frac{1}{4\pi} \iiint \frac{1}{r} \left(\frac{d\overline{F}_0}{dx} + \frac{d\overline{G}_0}{dy} + \frac{d\overline{H}_0}{dz} \right) dxdydz,$$

we have, by Poisson's equation,

$$\overline{F}_0 = \frac{1}{4\pi} \iiint \frac{F_2}{r} \, dx'dy'dz' - \frac{dX_0}{dx},$$

$$\overline{G}_0 = \frac{1}{4\pi} \iiint \frac{G_2}{r} \, dx'dy'dz' - \frac{dX_0}{dy},$$

$$\overline{H}_0 = \frac{1}{4\pi} \iiint \frac{H_2}{r} \, dx'dy'dz' - \frac{dX_0}{dz},$$

and, in general,

$$F_n = \frac{1}{4\pi} \iiint \frac{F_{n+2}}{r} \, dx'dy'dz',$$

$$G_n = \frac{1}{4\pi} \iiint \frac{G_{n+2}}{r} \, dx'dy'dz',$$

$$H_n = \frac{1}{4\pi} \iiint \frac{H_{n+2}}{r} \, dx'dy'dz'.$$

From these, by differentiation, we have, after changing $n+1$ to n,

$$F_n = \frac{1}{4\pi} \iiint \left\{ H_{n+1} \frac{d\frac{1}{r}}{dy} - G_{n+1} \frac{d\frac{1}{r}}{dz} \right\} dx'dy'dz',$$

$$G_n = \frac{1}{4\pi} \iiint \left\{ F_{n+1} \frac{d\frac{1}{r}}{dz} - H_{n+1} \frac{d\frac{1}{r}}{dx} \right\} dx'dy'dz',$$

$$H_n = \frac{1}{4\pi} \iiint \left\{ G_{n+1} \frac{d\frac{1}{r}}{dx} - F_{n+1} \frac{d\frac{1}{r}}{dy} \right\} dx'dy'dz'.$$

Let us now write

$$F_n = F_n + \frac{d\chi_n}{dx},$$

$$G_n = \bar{G}_n + \frac{d\chi_n}{dy},$$

$$H_n = \bar{H}_n + \frac{d\chi_n}{dz},$$

where

$$\chi_n = \frac{1}{4\pi}\iiint \frac{1}{r}\left\{\frac{d\bar{F}_n}{dx} + \frac{d\bar{G}_n}{dy} + \frac{d\bar{H}_n}{dz}\right\} dx'dy'dz'.$$

For all points where $\bar{F}_n' = \bar{G}_n = \bar{H}_n = 0$, $-\chi_n$ will be the potential of F_n, G_n, H_n. From the form of these equations, F_n, G_n, and H_n satisfy the equation of continuity, while $\bar{F}_n$, $\bar{G}_n$, and $\bar{H}_n$ do not. *Hence every discontinuous system of vectors, $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$, can be made continuous by the addition of the derivatives of a certain quantity having the value*

$$\frac{1}{4\pi}\iiint \frac{1}{r}\left\{\frac{d\bar{F}}{dx} + \frac{d\bar{G}}{dy} + \frac{d\bar{H}}{dz}\right\} dxdydz$$

and the new system so formed satisfies the equation of continuity.

To find a discontinuous system from a continuous one, we can draw surfaces which divide space into acyclic regions with respect to the kind of motion, and the discontinuous system will be distributed over these surfaces.

If there is some necessary physical condition by which the vectors *must* satisfy the equation of continuity, then the discontinuous* system of vectors $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$ will evidently have the same effect in every way as the continuous system F_n, G_n, and H_n, so that one system implies the existence of the other.

This theorem is at the basis of all theorems of such a nature as those which express the energy of a magnetic system either by an integral throughout space, or by one throughout the magnets, or by one over the surface of the magnets.

The value of χ_n can be expressed otherwise, as in Maxwell's Electricity, Art. 385, where it is shown that the integral can be divided into two integrals, one a surface integral and the other a volume integral. In comparing my formulæ with Maxwell's, however, it must not be forgotten that my volume integrals are to be taken throughout the *whole of space*, whereas Maxwell's gen-

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* In this paper I nearly always use the term "continuous system of vectors" to indicate a system which satisfies the equation of continuity.

But a distribution of vectors can evidently have a *side* discontinuity as well as an end discontinuity. The context will always indicate my meaning, especially as I use the dash over quantities which have *end* discontinuity and so do not satisfy the equation of continuity.

erally refer to the interior of some surface. Hence, in the case referred to, the surface is at an infinite distance and the two volume integrals are equal.

Consider the integral

$$\chi_n = \frac{1}{4\pi}\iiint \frac{1}{r}\left\{\frac{d\bar{F}_n}{dx} + \frac{d\bar{G}_n}{dy} + \frac{d\bar{H}_n}{dz}\right\} dxdydz$$

taken without exception throughout space.

This is equal to the sum of a surface and volume integral; but if we take the surface where $\bar{F}_n$, $\bar{G}_n$, and $\bar{H}_n$ are zero, the surface integral becomes zero, and leaves

$$\chi_n = -\frac{1}{4\pi}\iiint\left\{\bar{F}_n\frac{d\frac{1}{r}}{dx} + \bar{G}_n\frac{d\frac{1}{r}}{dy} + \bar{H}_n\frac{d\frac{1}{r}}{dz}\right\} dxdydz,$$

or we may write it

$$\chi_n = -\frac{1}{4\pi}\iiint\frac{\bar{M}_n\cos\epsilon}{r^2}\, dxdydz,$$

where $\bar{M}_n$ is the resultant vector of the n^{th} order, and ϵ is the angle between it and r.

If the point is within the region where $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$ exist, the integral will vanish at the lower limit, $r = 0$, and so the value can still be used within such space. For a region within which

$$\frac{d\bar{F}_n}{dx} + \frac{d\bar{G}_n}{dy} + \frac{d\bar{H}_n}{dz} = 0$$

the integral is much simplified. For it is only at the surface that this quantity has a value. Let the surface consist of an infinitely thin region of thickness $d\nu$, within which $\bar{M}_n$ decreases uniformly from its value inside to zero, then

$$\frac{d\bar{F}_n}{dx} = -\frac{d\bar{F}_n}{d\nu}\frac{d\nu}{dx},$$

$$\frac{d\bar{G}_n}{dy} = -\frac{d\bar{G}_n}{d\nu}\frac{d\nu}{dy},$$

$$\frac{d\bar{H}_n}{dz} = -\frac{d\bar{H}_n}{d\nu}\frac{d\nu}{dz}.$$

And since $dxdydz = d\nu dS$, we have

$$\chi_n = -\frac{1}{4\pi}\iint\frac{\bar{\phi}_n\cos\theta}{r^2}\, dS,$$

where $\bar{\phi}_n$ is the potential of $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$, and θ is the angle between the radius vector, r, and the normal drawn outwards from the surface, and the inte-

gration is over the surface. Multiplying the equations by $\frac{1}{r}\,dxdydz$, and integrating, we see that we can write

$$F_n = \frac{1}{4\pi}\iiint \frac{\overline{F}_{n+2}}{r}\,dx'dy'dz' - \frac{d\xi_{n+2}}{dx},$$

$$G_n = \frac{1}{4\pi}\iiint \frac{\overline{G}_{n+2}}{r}\,dx'dy'dz' - \frac{d\xi_{n+2}}{dy},$$

$$H_n = \frac{1}{4\pi}\iiint \frac{\overline{H}_{n+2}}{r}\,dx'dy'dz' - \frac{d\xi_{n+2}}{dz}.$$

By differentiation and replacement of $n+1$ by n, we have

$$F_n = \frac{1}{4\pi}\iiint \left\{ \overline{H}_{n+1}\frac{d\frac{1}{r}}{dy} - \overline{G}_{n+1}\frac{d\frac{1}{r}}{dz} \right\} dxdydz - \frac{d}{dx}\left\{ \frac{d\xi_{n+1}}{dy} - \frac{d\xi_{n+1}}{dz} \right\},$$

$$G_n = \frac{1}{4\pi}\iiint \left\{ \overline{F}_{n+1}\frac{d\frac{1}{r}}{dz} - \overline{H}_{n+1}\frac{d\frac{1}{r}}{dx} \right\} dxdydz - \frac{d}{dy}\left\{ \frac{d\xi_{n+1}}{dz} - \frac{d\xi_{n+1}}{dx} \right\},$$

$$H_n = \frac{1}{4\pi}\iiint \left\{ \overline{G}_{n+1}\frac{d\frac{1}{r}}{dx} - \overline{F}_{n+1}\frac{d\frac{1}{r}}{dy} \right\} dxdydz - \frac{d}{dz}\left\{ \frac{d\xi_{n+1}}{dx} - \frac{d\xi_{n+1}}{dy} \right\},$$

where the value of ξ_{n+2} is

$$\xi_{n+2} = -\frac{1}{4\pi}\iiint \frac{\chi_{n+2}}{r}\,dxdydz$$

or $$\xi_{n+2} = -\frac{1}{16\pi^2}\iiint\iiint \frac{1}{Rr}\left\{ \frac{d\overline{F}_{n+2}}{dx} + \frac{d\overline{G}_{n+2}}{dy} + \frac{d\overline{H}_{n+2}}{dz} \right\} dxdydz\,dx'dy'dz'.$$

One of these integrations is to be taken throughout all space, supposing $\overline{F}_{n+2}$, $\overline{G}_{n+2}$, $\overline{H}_{n+2}$ constant, and the other throughout space, supposing them variable.

The first can be performed, for it is simply

$$\iiint \frac{1}{Rr}\,dxdydz;$$

but we know that

$$\Delta^2 r = \frac{2}{r},$$

whence we may write

$$\iiint \frac{1}{Rr}\,dxdydz = -2\pi r.$$

These letters are connected by the relation that the R and r of the left-hand member and the r of the right-hand one form a triangle, the element $dxdydz$ being at the intersection of the R and r of the left-hand member. The integral vanishes for the limit $R = 0$.

Hence $$\xi_{n+2}=\frac{1}{8\pi}\iiint r\left\{\frac{d\bar{F}_{n+2}}{dx'}+\frac{d\bar{G}_{n+2}}{dy'}+\frac{d\bar{H}_{n+2}}{dz'}\right\}dx'dy'dz'.$$

This can be put in the form

$$\xi_{n+2}=-\frac{1}{8\pi}\iiint \bar{M}_{n+2}\cos\epsilon\, dxdydz,$$

where ϵ is the angle between $\bar{M}_n$ and the radius vector r.

We can now develop the following general method of finding the lower vectors from the higher. Let us write

$$O_n=\frac{1}{2}\iiint \bar{F}_n r dx'dy'dz',$$

$$P_n=\frac{1}{2}\iiint \bar{G}_n r dx'dy'dz',$$

$$Q_n=\frac{1}{2}\iiint \bar{H}_n r dx'dy'dz'.$$

Then we can write

$$\xi_n=-\frac{1}{4\pi}\left(\frac{dO_n}{dx}+\frac{dP_n}{dy}+\frac{dQ_n}{dz}\right)=\frac{1}{4\pi}\left(\frac{dO_n}{dx'}+\frac{dP_n}{dy'}+\frac{dQ_n}{dz'}\right),$$

and the equations become

$$F_{n-2}=\frac{1}{4\pi}\left\{\Delta^2 O_n+\frac{d}{dx}\left(\frac{dO_n}{dx}+\frac{dP_n}{dy}+\frac{dQ_n}{dz}\right)\right\},$$

$$G_{n-2}=\frac{1}{4\pi}\left\{\Delta^2 P_n+\frac{d}{dy}\left(\frac{dO_n}{dx}+\frac{dP_n}{dy}+\frac{dQ_n}{dz}\right)\right\},$$

$$H_{n-2}=\frac{1}{4\pi}\left\{\Delta^2 Q_n+\frac{d}{dz}\left(\frac{dO_n}{dx}+\frac{dP_n}{dy}+\frac{dQ_n}{dz}\right)\right\}.$$

Let us now see what will happen if we multiply these by $\frac{1}{4\pi R}dxdydz$ and integrate throughout space, R being a radius vector.

The first terms become F_{n-4}, G_{n-4}, and H_{n-4}, and we only have to consider the integrals of O_n, P_n, Q_n. We have to obtain an integral of the form

$$\iiint \frac{f(x'y'z')\,\phi(r)}{R}\,dxdydz\,,\,dx'dy'dz'.$$

Here we note that the radius vector r is the distance between the two points x, y, z and x', y', z', and so the integral reduces to

$$\iiint f(x'y'z')\iiint\left\{\frac{\phi(r)}{R}\,dxdydz\right\}dx'dy'dz'.$$

In this case, to obtain the integral with respect to r we make use of the fact that

$$\Delta^2 r^n = n(n+1) r^{n-2},$$

and we thus find

$$\frac{1}{4\pi} \iiint \frac{r^n}{R}\, dxdydz = -\frac{1}{(n+2)(n+3)}\, r^{n+2}.$$

The integral extends throughout space, and the r and R of the first member, and the r of the second member form a triangle, the point x, y, z being at their intersection. Hence,

$$\Delta^2 \iiint f(x', y', z')\, r^n dx'dy'dz' = n(n+1) \iiint f(x', y', z')\, r^{n-2} dx'dy'dz'.$$

We have thus found F_{n-4}, G_{n-4}, H_{n-4}, and by repeating the operation s times, counting the original operation as one, we can find F_{n-2s}, G_{n-2s}, H_{n-2s}, and F_{n-2s+1}, G_{n-2s+1}, H_{n-2s+1} by differentiation of these. The result is

$$O_{n,s} = \frac{(-1)^{s+1}}{1, 2, 3 \ldots . 2s} \iiint \bar{F}_n r^{2s-1}\, dxdydz,$$

$$P_{n,s} = \frac{(-1)^{s+1}}{1, 2, 3 \ldots . 2s} \iiint \bar{G}_n r^{2s-1}\, dxdydz,$$

$$Q_{n,s} = \frac{(-1)^{s+1}}{1, 2, 3 \ldots . 2s} \iiint \bar{H}_n r^{2s-1}\, dxdydz.$$

For the case $s = 0$, the coefficient of the integral becomes simply -1 and $\xi_{n,o}$ becomes $-\chi_n$. We also observe that

$$O_{n(s-t)} = (\Delta^2)^t (O_{n,s}),$$

$$P_{n(s-t)} = (\Delta^2)^t (P_{n,s}),$$

$$Q_{n(s-t)} = (\Delta^2)^t (Q_{n,s}),$$

and

$$\xi_{n,s} = -\frac{1}{4\pi} \left(\frac{dO_{n,s}}{dx} + \frac{dP_{n,s}}{dy} + \frac{dQ_{n,s}}{dz} \right),$$

$$\xi_{n(s-t)} = (\Delta^2)^t \xi_{n,s}.$$

In these the symbol $(\Delta^2)^t$ signifies that the operation Δ^2 is to be repeated t times. Hence we may write, in general,

$$F_{n-2s} = \frac{O_{n,(s-1)}}{4\pi} - \frac{d\xi_{n,s}}{dx},$$

$$G_{n-2s} = \frac{P_{n,(s-1)}}{4\pi} - \frac{d\xi_{n,s}}{dy},$$

$$H_{n-2s} = \frac{Q_{n,(s-1)}}{4\pi} - \frac{d\xi_{n,s}}{dz.},$$

and

$$F_{n-2s+1} = \frac{dH_{n-2s}}{dy} - \frac{dG_{n-2s}}{dz},$$

$$G_{n-2s+1} = \frac{dF_{n-2s}}{dz} - \frac{dH_{n-2s}}{dx},$$

$$H_{n-2s+1} = \frac{dG_{n-2s}}{dx} - \frac{nF_{n-2s}}{dy}.$$

Thus we have expressed the whole series of vectors in terms of any one series, and have shown how to calculate the distribution of them all from the discontinuous distribution of any one order, and this by a single set of integrals, with the proper differentiation.

The value of $\xi_{n,s}$ can be expressed simply by the integral

$$\xi_{n,s} = \frac{1}{4\pi}\frac{(-1)^{s+1}}{1,2,3\ldots\ldots 2s}\iiint \bar{M}_n r^{2s-2}\cos\epsilon\, dxdydz,$$

where ϵ is the angle between $\bar{M}_n$ and r. For all space in which $\bar{M}_n$ is zero, F_n, G_n, H_n have a potential found by making $s = 0$, whence

$$\xi_{n.0} = -\chi_n = \frac{1}{4\pi}\iiint \frac{\bar{M}_n \cos\epsilon}{r^2}\, dxdydz.$$

$\xi_{n,0}$ satisfies Laplace's equation at all points where $\bar{M}_n = 0$, and this potential can be considered as due either to the distribution of $\bar{M}_n$ within a surface, or of M_{n+1} over the surface, the components of M_{n+1} satisfying the equation of continuity, but those of $\bar{M}_n$ not.

Indeed, we can consider *any* of the systems of vectors as the *cause* and all the others as the *effect*.

When $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$ satisfy the equation of continuity within a surface, the integrals reduce to the surface integral of the next higher order of vectors.

But in the above operation of finding F_{n-2s}, G_{n-2s}, H_{n-2s} from $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$, we have neglected the constants of integration, and we have now to consider their effect.

We first observe that, as the higher orders are obtained from the lower by differentiating, they are perfectly determined by these equations. But as we

have to integrate in obtaining the lower orders from the higher, the result requires examination. Taking the equations for F_n, G_n, H_n, and adding to them respectively π, π_1, π_{11}, we have

$$F_n = \frac{1}{4\pi}\iiint \frac{F_{n+2}}{r}\, dx'dy'dz' + \pi,$$

$$G_n = \frac{1}{4\pi}\iiint \frac{G_{n+2}}{r}\, dx'dy'dz' + \pi_1,$$

$$H_n = \frac{1}{4\pi}\iiint \frac{H_{n+2}}{r}\, dx'dy'dz' + \pi_{11},$$

Performing the operation Δ^2 on these, with respect to the point x', y', z', and also noting that F_n, G_n, H_n satisfy the equation of continuity, we shall have

$$\Delta^2\pi = 0, \qquad \Delta^2\pi_1 = 0, \qquad \Delta^2\pi_{11} = 0,$$

$$\frac{d\pi}{dx} + \frac{d\pi_1}{dy} + \frac{d\pi_{11}}{dz} = 0.$$

Now these quantities must not affect the values of F_{n+1}, G_{n+1}, H_{n+1} as obtained from F_n, G_n, H_n, and so we must also have

$$\frac{d\pi_{11}}{dy} - \frac{d\pi_1}{dz} = 0,$$

$$\frac{d\pi}{dz} - \frac{d\pi_{11}}{dx} = 0,$$

$$\frac{d\pi_1}{dx} - \frac{d\pi}{dy} = 0.$$

The solution of these equations is found in the values

$$\pi = \frac{dV}{dx}, \qquad \pi_1 = \frac{dV}{dy}, \qquad \pi_{11} = \frac{dV}{dz},$$

where V must satisfy Laplace's equation throughout all space. And a term of this kind is the proper thing to add, no matter how many integrations are performed. But if this term satisfies Laplace's equation throughout all space, the *cause* of the potential V must be at an infinite distance. As this quantity is entirely arbitrary, and has no relation to the original system of vectors which we consider as the *cause* of the vectors of the lower order, we may generally make it zero unless there is some other condition to be satisfied. But the integrals without any addition give us all the lower orders of vectors which are dependent on the higher. Hence no addition should be made to the integrals when we are searching for the effect of a cause.

Indeed, all such cases of a potential satisfying Laplace's equation throughout space can always be represented by a distribution over a sphere at an infinite distance, and in Physics it must always be looked upon in that way. In fluid motion the only quantities which we can add without destroying the original configuration is a general motion of translation and rotation of the fluid represented by the equations

$$\pi = A + Bx + Cy + Dz,$$

$$\pi_1 = A_1 + B_1x + C_1y + D_1z,$$

$$\pi_{11} = A_{11} + B_{11}x + C_{11}y + D_{11}z,$$

where we have the relations

$$B + C_1 + D_{11} = 0; \;\; C_{11} = D_1; \;\; D = B_{11}; \;\; C = B_1.$$

We can define V by saying that it is a solid harmonic of a positive degree, so that it becomes infinite at an infinite distance.

We see from these facts that if any quantities, F_n, G_n, H_n, vanish, then all the others must have the values given by the above equations.

Hence all the systems of vectors must exist somewhere in space, though it is evident that there may be regions where some of the orders vanish. When this is the case, and throughout the region the vectors of the n^{th} order vanish, then all orders of vectors greater than the n^{th} vanish also, throughout the same space, but not those less than this; and the vectors of the $(n-1)^{\text{th}}$ order have a potential satisfying Laplace's equation.

Furthermore, if within any surface we have a discontinuous distribution of the n^{th} order of vectors, then we have seen that without the surface the vectors F_n, G_n, H_n have a potential satisfying Laplace's equation, and consequently all vectors above the n^{th} order will vanish outside that surface.

If within or over a given surface the vectors $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$ form closed circuits, then the vectors F_n, G_n, H_n, and of higher orders due to the above vectors will be zero without the surface, and the vectors of the $(n-1)^{\text{th}}$ order will have a potential. But in the surface, or without it, all the vectors exist. Conversely, if throughout any space the vectors of the n^{th} order have a potential satisfying Laplace's equation, the vectors of the $(n-1)^{\text{th}}$ order will be zero in that space.

If within any surface we have a discontinuous distribution of the n^{th} order of vectors, it is evident that the external effect will be the same as the continuous distribution within the same surface of the $(n-1)^{\text{th}}$ or higher orders. Indeed, the effect will be the same throughout space.

We have seen that there are two methods of looking at a series of vectors, — one as if they were derived from those above, and the other from those below. Now these methods are similar to the direct and inverse methods of treating electrical problems. The inverse method invented by Green is by far the most useful in electricity, and it seems to me that a similar method will be useful in the treatment of vector quantities. Indeed, the method leads to most important results. Consider the equations

$$F_{n+1} = \frac{dH_n}{dy} - \frac{dG_n}{dz},$$

$$G_{n+1} = \frac{dF_n}{dz} - \frac{dH}{dx},$$

$$H_{n+1} = \frac{dG_n}{dx} - \frac{dF}{dy}.$$

The operations here indicate that we choose the vectors of the n^{th} order, and find their *cause* in the vectors of the $(n+1)^{\text{th}}$ order; whereas the direct operation assumes the *cause* and finds the *effect*. This is similar to the case of electricity where the *cause* of the potential V is discovered by the operation $-\frac{1}{4\pi}\Delta^2$ to be the density of electricity ρ, with its proper distribution. And just as we find many electrical distributions over surfaces which have the same external effect, so we are able to find many distributions of the vectors F_{n+1}, G_{n+1}, H_{n+1}, over surfaces which produce the same external values of F_n, G_n, and H_n.

Suppose that within a surface F_n, G_n, H_n have a potential $\xi_{n,0}$, and outside a potential $\xi'_{n,0}$, both of which satisfy Laplace's equation. To satisfy the equation of continuity we must have at the surface,

$$\frac{d\xi_{n,0}}{d\nu} = \frac{d\xi'_{n,0}}{d\nu}.$$

Our equations then show that F_{n+1}, G_{n+1}, H_{n+1} are zero throughout all space except the surface. At the surface the equations become

$$F_{n+1}\,d\nu = \left(\frac{d\xi'_{n,0}}{dz} - \frac{d\xi_{n,0}}{dz}\right)\frac{d\nu}{dy} - \left(\frac{d\xi'_{n,0}}{dy} - \frac{d\xi_{n,0}}{dy}\right)\frac{d\nu}{dz},$$

$$G_{n+1}\,d\nu = \left(\frac{d\xi'_{n,0}}{dx} - \frac{d\xi_{n,0}}{dx}\right)\frac{d\nu}{dz} - \left(\frac{d\xi'_{n,0}}{dz} - \frac{d\xi_{n,0}}{dz}\right)\frac{d\nu}{dx},$$

$$H_{n+1}\,d\nu = \left(\frac{d\xi'_{n,0}}{dy} - \frac{d\xi_{n,0}}{dy}\right)\frac{d\nu}{dx} - \left(\frac{d\xi'_{n,0}}{dx} - \frac{d\xi_{n,0}}{dx}\right)\frac{d\nu}{dy}.$$

This distribution of F_{n+1}, G_{n+1}, H_{n+1} will produce the potentials $\xi_{n,0}$ and $\xi'_{n,0}$, the one within and the other without the surface. The most interesting case of this discontinuity at a surface is found by making

$$\frac{d\xi_{n,0}}{d\nu} = 0 \quad \text{and} \quad \frac{d\xi'_{n,0}}{d\nu} = 0$$

at the surface, in which case the surface is a stream surface for both $\xi'_{n,0}$ and $\xi_{n,0}$, and the equation of continuity for the n^{th} order is satisfied.

Let us now make $\xi_{n,0}$ = constant, and we have

$$F_{n+1}\,d\nu = \frac{d\xi'_{n,0}}{dz}\frac{d\nu}{dy} - \frac{d\xi'_{n,0}}{dy}\frac{d\nu}{dz},$$

$$G_{n+1}\,d\nu = \frac{d\xi'_{n,0}}{dx}\frac{d\nu}{dz} - \frac{d\xi'_{n,0}}{dz}\frac{d\nu}{dx},$$

$$H_{n+1}\,d\nu = \frac{d\xi'}{dy}\frac{d\nu}{dx} - \frac{d\xi'_{n,0}}{dx}\frac{d\nu}{dy},$$

which will apply to all surfaces as well as stream surfaces.

If we had made $\xi'_{n,0}$ = constant instead of $\xi_{n,0}$, we should have obtained equations of the same form, but with the opposite sign. If, however, we say that the normal shall be drawn from the side of the surface containing the original source of potential to the other side, the sign will be the same. Thus our surface must be drawn so as to separate space into two parts, one of which contains the source, and our equations give us the values to distribute over the surface, so that, on the side *opposite* the source, the effect will be the same as from the original source. Hence, although we may express a proposition with respect to the *outside* and *inside*, yet we may always reverse the terms into *inside* and *outside*. This is exactly analogous to the proposition with regard to the distribution of electricity. If we distribute the vectors over the surface with a negative sign, they will just neutralize the effect of the original source on the side opposite to it, which is exactly analogous to the case of electrostatic induction.

As to the total amount of the vector M_{n+1} to be distributed over the surface, it is evident that the *quantity* in the two systems must be the same.

I here use the term quantity in the sense of surface integral across the section of the vector. Hence, if M'_{n+1} is the original and M_{n+1} the new value of the vector, we must have

$$\iint M'_{n+1}\,dS' = \iint M_{n+1}\,dS,$$

the integral to be taken over a surface cutting the vectors at right angles.

Having chosen our stream surface of the n^{th} order, we now have to define the direction of M_{n+1} on the surface. The equations express the fact that $M_{n+1}d\nu$ is in the direction of the intersection of the given surface with the equipotential surface, and has the value M_n. The equipotential surfaces and two systems of stream surfaces can be found so as to form an orthogonal system. But a stream surface in general is not necessarily one of them, but is only a surface containing the stream lines or lines of direction of M_n. When we choose for our surface one of the stream surfaces of the orthogonal system, the intersection of the two stream surfaces is an equipotential line for M_{n+1}, and the intersection of the stream surface and an equipotential surface is a stream line for M_{n+1}. We have seen that the potential ξ_n' of the n^{th} order can arise either from a distribution of vectors of the $(n+1)^{\text{th}}$ order in closed circuits or a discontinuous distribution of the n^{th} order; and that the $(n+1)^{\text{th}}$ order was distributed in closed lines around the n^{th} order. The relations are similar to those between magnetism and electricity and between a magnetic shell and a current around its edge. In this case we can obtain a distribution of a shell of $\bar{M}_n$ over the surface which is equivalent to the distribution of M_{n+1} over the surface. The strength of the shell, $\bar{M}_n d\nu$, will evidently be found by adding all the shells together, and hence, if we draw any line l on the surface,

$$\bar{M}_n d\nu = \int M_{n+1} d\nu \cos\theta \, dl,$$

but this is equivalent to

$$\bar{M}_n d\nu = \int M_n d\nu \cos\theta \, dl = \xi_{n,0} + \text{constant},$$

because $M_{n+1}d\nu$ is equal to M_n at the surface.

Hence the strength of the shell is equal to the potential. Knowing $M_{n+1}d\nu$ originally, we can thus find $\bar{M}_n d\nu$, whence the potential is

$$\xi_{n,0}' = \frac{1}{4\pi} \iint \frac{\bar{M}_n d\nu \cos\epsilon}{r^2} \, dS,$$

ϵ being the angle between r and the *normal* to the surface. This integral must not only be taken over the surface which we have been considering, but also over diaphragms which we must draw so as to convert cyclic space into acyclic.

Thus the theorem which I have just given merely shows us how to draw surfaces so that this integral taken over them shall be the same for all regions which can be reached without passing through a surface.

It has already been known that the diaphragms could be drawn in any position, but this theorem applies also to the other positions of the surface boundary of the space under consideration.

We may evidently have two varieties of surfaces which divide the internal space into cyclic or acyclic regions. The first are obtained by forming, as it were, tubes around the closed circuits of the vector M_{n+1}, which we can consider as the original source of $\xi'_{n,0}$. Distributing M_n over this surface in the manner described, and removing the original source, the new system has the same external effect as the old, but makes $\xi_{n,0}$ constant inside. The second variety of surface is acyclic inside, and encloses the original distribution of M_{n+1} as a whole. It is evident that this last variety of surface cannot be a stream surface for $\xi'_{n,0}$, unless there is some other source of M_n outside the surface. Our process, then, merely replaces one of these sources for the points on the other side of the surface.

Taking any cyclic value of the potential $\xi'_{n,0}$ and drawing a tube of flow of any cross section, and applying this method to it, the distribution of M_{n+1} over the surface will cause the potential inside it to become constant. Take away the external system now, and we shall have a potential equal to $-\xi'_{n,0}$ inside the tube, and constant without.

If we take any two of our surfaces and distribute $+M_{n+1}$ over the internal one, and $-M_{n+1}$ over the external one, according to the equations, then within the inside surface we shall have $\xi_{n,0}$ constant; between the two surfaces $\xi'_{n,0}$ has its former value; and outside the outer one $\xi_{n,0}$ is constant once more. In fluid motion this will give us the case of a mass of liquid revolving in a liquid at rest, with a core of liquid at rest.

If we allow the original distribution to remain, and distribute $-M_{n+1}$ over the surface, then the potential *inside* remains the same and becomes constant *outside*. Thus the inside and outside of the surface are reciprocal, and we can always reverse the statement in this manner.

Helmholtz has considered some cases of discontinuous fluid motion, and other writers have occupied themselves more or less with it. But most, if not all of them have considered the vortex surface between the two moving fluids as the *effect*; whereas I here treat the vortices as the *cause* of the fluid motion, and have thus obtained a method of replacing the original vortices by one or more vortex sheets around them. This is the basis of the inverse method of treating hydrodynamics.

To get a solid distribution of M_{n+1}, we can place one surface within the other with its proper distribution so as to make what is required. If within a given surface any system of vectors are distributed so as to have a potential within that surface, then the continuous system as obtained from this system will have a potential both within and without the surface, but the potential inside will not

be the same as that of the discontinuous system, but will be that minus $\xi_{n,0}$. At the surface of discontinuity all the higher system of vectors will be collected.

Let us suppose that we have the potential $\xi_{n,0}$ within the surface, and $\xi'_{n,s}$ outside the surface, both of which satisfy Laplace's equation for the space in which they are used. We can get the conditions at any surface, but as they are complicated, let us take the surfaces, stream surfaces for both functions.

Then at the surface

$$\frac{d\xi_{n,0}}{dv} = 0; \quad \frac{d\xi'_{n,s}}{dv} = 0.$$

According to our notation, $\xi_{n,0}$ is the potential of a *higher* order of motion than $\xi'_{n,s}$, the first being the potential of F_n, G_n, H_n and the last of F_{n-2s}, G_{n-2s}, H_{n-2s}.

If we should distribute $-F_{n+1}$, $-G_{n+1}$, $-H_{n+1}$, over the surface according to the previous investigation, we should have $\xi_{n,0} = \xi_{n,0}$ inside, and constant outside. If F_{n-2s+1}, G_{n-2s+1}, H_{n-2s+1}, were distributed over the surface in the proper manner, we should have $\xi'_{n,s} = \xi'_{n,s}$ outside and constant inside. If the two existed together, there would be

$$\xi_{n,0} + \text{constant inside,}$$

and

$$\xi'_{n,s} + \text{constant outside.}$$

So that in this case we must distribute over the surface

$$-F_{n+1}\,d\nu = -\frac{d\xi_{n,0}}{dz}\frac{d\nu}{dy} + \frac{d\xi_{n,0}}{dy}\frac{d\nu}{dz},$$

$$-G_{n+1}\,d\nu = -\frac{d\xi_{n,0}}{dx}\frac{d\nu}{dz} + \frac{d\xi_{n,0}}{dz}\frac{d\nu}{dx},$$

$$-H_{n+1}\,d\nu = -\frac{d\xi_{n,0}}{dy}\frac{d\nu}{dx} + \frac{d\xi_{n,0}}{dx}\frac{d\nu}{dy},$$

and

$$F_{n-2s+1}\,d\nu = \frac{d\xi'_{n,s}}{dz}\frac{d\nu}{dy} - \frac{d\xi'_{n,s}}{dy}\frac{d\nu}{dz},$$

$$G_{n-2s+1}\,d\nu = \frac{d\xi'_{n,s}}{dx}\frac{d\nu}{dz} - \frac{d\xi'_{n,s}}{dz}\frac{d\nu}{dx},$$

$$H_{n-2s+1}\,d\nu = \frac{d\xi'_{n,s}}{dy}\frac{d\nu}{dx} - \frac{d\xi'_{n,s}}{dx}\frac{d\nu}{dy}.$$

Gauss's theorem applied to electricity gives us the amount of electricity in a surface, from the surface integral of the electric force. Let us now attempt to develop some similar method for this case.

Taking Poisson's equation,

$$4\pi\rho = -\Delta^2 V,$$

Gauss's theorem gives

$$\iint \frac{dV}{dv}\, dS = -4\pi \iiint \rho\, dx dy dz,$$

where the first integral extends over a surface, and the second over the space included within the surface. If we have a distribution of F_n, G_n, H_n, within a given surface, then F_{n+2}, G_{n+2}, H_{n+2}, are obtained from them by the equations

$$F_{n+2} = -\Delta^2 F_n,$$

$$G_{n+2} = -\Delta^2 G_n,$$

$$H_{n+2} = -\Delta^2 H_n.$$

Hence we can write immediately

$$\iint \frac{dF_n}{dv}\, dS = -\iiint F_{n+2} dx dy dz,$$

$$\iint \frac{dG_n}{dv}\, dS = -\iiint G_{n+2} dx dy dz,$$

$$\iint \frac{dH_n}{dv}\, dS = -\iiint H_{n+2} dx dy dz,$$

As we have

$$J_n = -\Delta^2 \chi_n = \Delta^2 \xi_{n,0},$$

therefore we can also write

$$\iint \frac{d\xi_{n,0}}{dv}\, dS = -\iiint J_n dx dy dz,$$

where

$$J_n = \frac{d\bar{F}_n}{dx} + \frac{d\bar{G}_n}{dy} + \frac{d\bar{H}_n}{dz}.$$

To get a clear idea of the meaning of J_n, I may remark that in the case of magnetism it gives us the distribution of the so called magnetic matter. In the present case the integral vanishes over any surface enclosing the original source, so that the algebraical sum of J_n within the surface is zero, or the poles of the magnet are of equal strength.

I have thus developed the general theory of vectors from a purely mathematical point of view, so that we may apply it to any subject involving these important quantities, and thus many excellent methods for use in electromagnetism may be found. There are even some methods which give important

results in the theory of attraction and spherical harmonics, especially the theorem that

$$\Delta^2 \iiint f(x', y', z',)\, r^n dx' dy' dz' = n(n+1) \iiint f(x', y', z',)\, r^{n-2} dx' dy' dz',$$

Δ^2 being taken with respect to x, y, z, and not with respect to x', y', z'. This gives the potential with a force varying as r^{n-3} from that varying as r^{n-1}.

But the most important and interesting application is to hydrodynamics, and I shall then develop a few more points which might have been placed above.

There is an important point connected with this theory which will be extremely useful to us further on; and that is that where M_n exists and varies from one point to another, then we shall sometimes be obliged to use an element of the fluid in our calculations for the determination of M_0 infinitely smaller than for M_n. In other words, M_0 will not be determined exactly within an element without going still lower and assuming something further within the element. Thus, in case we have an infinitely small tube having the vector M_n in the direction of its length, we find that there are an infinite number of ways in which the vector can be distributed within the tube and yet produce the same external effect. This may be unimportant in some cases, but a neglect of this principle will cause many interesting problems in hydrodynamics to assume an indeterminate form. Thus the problem of the velocity of a vortex ring needs to be calculated in this way. Assuming the distribution of finer vortex filaments within the substance of the vortex, and the problem becomes determinate. I shall thus show that the vortex ring moves according to that distribution.

The Kinematics of Fluid Motion.

The fluid is considered as unlimited and without solid bodies in it. The motion of a fluid is a vector quantity, and so the above theory must apply.

Let F_0, G_0, H_0, be the components of the fluid velocity,* and let the fluid be incompressible. Then we have

$$\frac{dF_0}{dx} + \frac{dG_0}{dy} + \frac{dH_0}{dz} = 0.$$

The vectors of the higher order, which I shall for simplicity call motions, must then be interpreted. The motion of the first order whose components are

* We could evidently take any other order for the fluid velocity, and, in any problem, we can generally change the suffixes to the letters, and thus obtain new solutions.

F_1, G_1, H_1, has been interpreted by Helmholtz to be vortex motion, and it remains to interpret the rest.

The second order will be, as it were, the vortices of the vortices, but to interpret it better I shall proceed as follows: —

Suppose that along the axis of x there is a line of motion of the n^{th} order. Calculate from this the values of F_0, G_0, and H_0, and we shall have the true motion of the fluid. It will be found that in all motions of the even degrees the fluid moves parallel to the axis of x, but in motions of the odd degree the true motion of the fluid is in circles around the axis. Hence the motions of the even degrees are translatory motions and those of the odd degrees are vortical motions.

Motion of the zero order is ordinarily irrotational motion.

Motion of the first order is the vortex motion of Helmholtz, and a line of motion of the first order is a vortex filament.

Motion of the second order is what I have, in a previous paper, called the *relative* motion of the fluid, and a line of this motion has the fluid flowing forward along its axis with very great velocity and a less and less velocity as we proceed outward from the axis, until we reach a certain distance, where it is zero; beyond this the velocity increases as we pass from the axis, and is infinite at an infinite distance in the impossible case of the line of motion being straight and infinitely long. In the possible cases where the line is closed, the motion of the fluid is finite everywhere, and is zero at infinity.

Motion of the third order can be roughly conceived of as one vortex filament within another revolving in the opposite direction.

Motion of the fourth order can be roughly conceived of as one line of motion of the second order within another larger and opposite one.

And so, as we go upward, an element of the line of motion becomes more and more complicated. The motion of the fluid exterior to the line of motion of any order can be calculated by formulæ as simple for high as for low orders of motion, though the integrations are generally somewhat more difficult as we go higher.

Mathematically there are motions of the negative orders also, and they are useful in some calculations; but the physical conception does not at first sight lead to such important results as the positive, though I hope to investigate this further.

Let us now conceive of a closed line in space of any form whatever having a uniform distribution of one of the higher motions along it.

As the curve is closed such a distribution will satisfy the equation of continuity.

If the motion along the line is of the n^{th} order, then there will be a distribution throughout space of motion of the $(n-1)^{th}$ order and below, but no motion of the n^{th} or higher orders. And the motion of the fluid in this case will be equivalent to that produced by a discontinuous but uniform distribution of motion of the $(n-1)^{th}$ order over any surface having its edge in the given line, the motion being perpendicular to the surface. Suppose now that the fluid in which this line is placed receives motion of the $(n-1)^{th}$ order, but no higher. In order that this may be possible, it must have a potential, and so will vanish *for this region* when we find the new values of the motion of the n^{th} order for this region. But not so if we add motion of the n^{th} order or higher.

Hence we have the theorem that motion of the n^{th} order is unaffected by motion of any lower order.

We have seen that if within a surface there are closed circuits of the n^{th} order of motion, then without the surface there is motion of the $(n-1)^{th}$ order and below it, the $(n-1)^{th}$ order having a scalar potential.

Hence, if two regions of this nature be placed near each other, the motion of the n^{th} and higher orders will be the same as before, but of the lower orders it will be changed.

In considering the general relations of vectors, I have virtually shown that if the fluid has any motion, then the whole series of motions must exist *somewhere* in space. But we have also seen that if one of the motions, say of the n^{th} order, is confined to any given region, then all the higher motions will also be confined to that region, and the remainder of the fluid will have only the $(n-1)^{th}$ order and below; and that the $(n-1)^{th}$ order outside a given region has a potential satisfying Laplace's equation, but the lower orders do not.

Furthermore, I have also shown how to find the motions of any order outside the given region by integrating throughout the given region.

Now suppose a given region has a distribution of motion of the n^{th} order within it which is discontinuous at the surface.

We have seen how to find the distribution of motion of the n^{th} order without it, in order to satisfy the equation of continuity. We have seen that this distribution has a potential without the region, and that only motion of the n^{th} order and below exists outside the region.

Now let us suppose that the original distribution of the n^{th} order within the region is also such that within that region a potential satisfying Laplace's equation exists, but not the same as that without. Then it is evident that all the motion of the $(n+1)^{th}$ and higher orders exists only at the surface of separation of the two regions, and that both within and without the region only the n^{th} and lower orders of motion exist.

Thus this surface constitutes a surface of discontinuity in the fluid motion of the n^{th} order.

Helmholtz has discussed the case of surfaces of discontinuity when the motion on the two sides was of the zero order, and thus the motion at the surfaces of the first and higher orders.

We have treated this case very fully in the theory of vectors, and have there shown the exact distribution of the motions.

The process deduced from this theory which is most useful in hydrodynamics is the following:—

Let us have a distribution of the $(n+1)^{\text{th}}$ in closed curves so that it satisfies the equation of continuity. Then throughout the rest of space the motion of the $(n+1)^{\text{th}}$ order will be zero, and the n^{th} order will have a cyclic potential $\xi_{n,0}$, the source of the fluid motion being supposed to be the $(n+1)^{\text{th}}$ motion. Select any stream surface which will thus enclose some of the $(n+1)^{\text{th}}$ motion.

Draw the equipotential surfaces for the n^{th} order of motion so as to intersect the given stream surface in lines, and distribute over the surface, in the direction of these lines, motion of the $(n+1)^{\text{th}}$ order, so that

$$M_{n+1}dv = M_n$$

at every point of the surface. Then the new distribution of the $(n+1)^{\text{th}}$ motion produces the same effect as the old. Therefore we can take away the old distribution within the surface and replace it by fluid for which the potential of the n^{th} order of motion is constant, and the external effect of the new system is the same as the old.

By putting one stream surface within the other a solid distribution of the $(n+1)^{\text{th}}$ order might be made. If we draw the equipotential surface in the manner that Maxwell has done, we can define the distribution of the $(n+1)^{\text{th}}$ order as follows:—

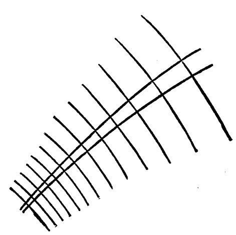

FIG. 1.

Let the figure represent a system of equipotential surfaces infinitely near together, intersected by two stream surfaces very near together.

Let $d\lambda$ be the distance apart of two equipotential surfaces. Then we can write

$$M_{n+1} d\nu d\lambda = M_n d\lambda.$$

But we also have

$$M_n = - \frac{d\xi_{n,0}}{d\lambda},^*$$

therefore we have

$$M_{n+1} d\nu d\lambda = - \frac{d\xi_{n,0}}{d\lambda} d\lambda.$$

But from the method of drawing the figure the last member is constant, and consequently the first is also. But the first is the total quantity of M_{n+1} within the rectangle, therefore the proper distribution of M_{n+1} is given by putting the same amount into each rectangle. Let us now attempt to get the total *quantity* of M_{n+1}. By *quantity* of M_{n+1} I mean the surface integral of its cross section. Thus, in this case, the quantity of M_{n+1} is

$$\int M_{n+1} d\nu d\lambda = - \int \frac{d\xi_{n,0}}{d\lambda} d\lambda,$$

the integrals to be taken around the section of the surface. The last integral is, in our notation, simply equal to the original *quantity* of $(n+1)^{\text{th}}$ motion. Hence, the quantity of $(n+1)^{\text{th}}$ motion distributed over the surface must be equal to the original quantity. The analogy of this to the case of electric distribution is apparent, the surface integral over the cross section of a vector taking the place of quantity of electricity.

If we apply this to the case of vortex and ordinary motion, we shall have the case of liquid at rest enclosed within a vortex surface, with the motion of the exterior liquid irrotational and expressed by a cyclic potential.

The vortices of the surface are constantly moving forward in the direction of the fluid motion immediately outside the surface.

If we wish a volume distribution of M_{n+1} within the surface, we merely have, as I have shown above, to put surfaces within one another with the above relative distribution, the strength of each surface being arbitrary. We have thus one other condition to fulfil arbitrarily. Let Q_n be a function which is constant for the stream surfaces, or, in other words, the stream function.

Then the solid distribution of M_{n+1} is represented by the formula

$$M_{n+1} = \frac{C}{d\nu dl} \psi(Q_n),$$

where C is a very minute constant, and $\psi(Q_n)$ is a function of Q_n.

* The notation here used is that of the general theory given on p. 236, so that $\xi_{n,0}$ is the same as $-\chi_n$ used before.

The direction of M_{n+1} is that of the intersection of the surfaces $Q_n =$ constant, and $\xi_{n,0} =$ constant.

Such a distribution of M_{n+1} will produce a potential outside the surface which will be proportional to $\xi_{n,0}$ and equal to it, if C has the proper value. But inside the surface the motion is dependent upon the form of $\psi(Q_n)$.

The distribution of the vortices over all the surfaces need not always be in the same direction, but we may alternate as often as we please without altering the outer distribution of motion.

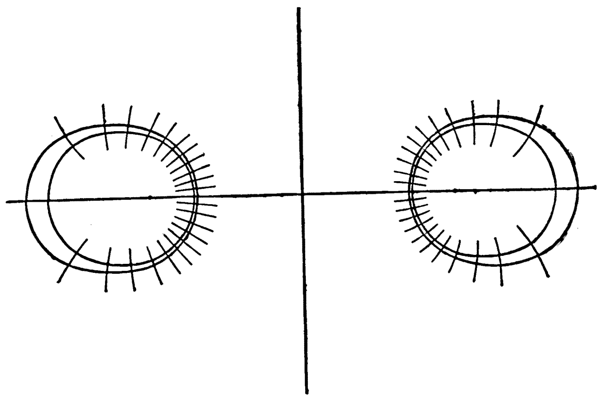

FIG. 2.

As an illustration let us take a circular vortex ring. The equipotential surfaces and lines of flow for an infinitely thin vortex are represented by the preceding diagram, where the closed curves of somewhat circular form are the sections of the surfaces of flow, and the others are the sections of the equipotential surfaces.

The motion of zero order is distributed throughout space according to the cyclic potential $\xi_{n,0}$, and the motion of the first order (vortex motion) is confined to the closed surface.

If we should cause instantaneous vortical forces to act over a surface similar to the given one, the forces being proportional to the vortices in the first case, then the analysis of vector motions shows that these forces produce the same effect as the original distribution of vortical forces for all points outside, but no effect on the points inside the surface. Hence, the system formed by these forces will be the same as the system under discussion. The system thus formed will undergo changes which are investigated in the portion of this paper treating of the equilibrium of fluid motion. We have there seen that the whole system of changes can be treated from the drifting of the vortex filaments, the whole

dynamics of the problem being satisfied by making the product of the vortex-strength of each filament by its cross-section constant as the element moves along. In the present case, the elementary vortex filaments of the vortex surface are constantly drifting from one place to another, and they may thus finally obtain another distribution, in which case the form of the surface and its position may change. In the present case, if the fluid moves through the ring from left to right, and we refer to the upper section, the vortex strength tends to increase on the outer and right-hand side, and decrease on the inner and left-hand side.

When there is any change of distribution, the ring as a whole will tend to revolve around the part which has increased the most, and so the whole ring tends, as a whole, to decrease in diameter and move forward, though the changes of form are not so obvious.

As the ring moves, the weaker part constantly tends to go inside, and the stronger part outside, and so tends to a form of equilibrium, though this should be investigated more thoroughly.

In treating a filament, we must descend to the elements of the filament itself and consider the motion of each secondary element as affected by the others. It is impossible to treat this filament or vortex surface as a whole. The fineness of filaments to which we must descend will depend upon the rate of variation of the vortex strength. Thus, when we are dealing with finite distributions of vortices without *side* discontinuity, the elementary filaments may be large, but when we treat filaments or vortex surfaces, the elementary filaments must be small in proportion.

If we now consider the hollow vortex to be very small in cross section, compared with the radius, so that it is nearly circular, and so does not change its shape much, it is easy to see that the tendency of the elementary vortices to accumulate on the outside will tend to revolve the whole circle into a position farther on in the direction of the fluid motion inside the ring, and if we still further consider that there is a tendency to accumulate in the front part also, we see that the ring will tend to move forward and to contract in radius. If the cross section is very small, the form of the cross section becomes more and more stable. The radius of the ring finally becomes constant when the forward motion is such as to make the difference of velocity between the fluid in the inside of the hollow vortex and the outside, the same for every portion of the cross section of the surface, which is possible for a very small cross section, or for a large one if the shape is correct.

In attempting to construct synthetically a hollow vortex which shall move

forward uniformly through space without change of form, we have simply to find a distribution of elementary vortex rings such that the fluid motion due to them combined with a uniform forward motion shall produce a stream surface along which the motion of the fluid is everywhere uniform as reckoned from a system of axes moving with the surface.

Or, for the sake of ease in calculating, we can suppose the axes at rest and the fluid flowing past them. It would thus be perfectly possible to construct synthetically a hollow vortex ring of finite cross section and finite velocity of translation.

In the case of a very thin, hollow vortex of circular cross section, the velocity can be obtained by finding the motion of the fluid in the interior due to all the vortex filaments of which the ring is formed. Or it is simply one half of the difference of velocity on the outside of the surface nearest and furthest from the axes of symmetry.

The velocity of translation of very thin solid vortex rings can readily be found by calculating the velocity at the core due to the given distribution of vortex filaments constituting the ring. It will thus be found that for a given distribution of vortex filaments within the very thin vortex, the vortex must move perpendicular to the plane of its curvature and inversely as the radius of curvature, to insure stability. Thus, if we curve a circular vortex at any point outward, at first the curved part will move faster than the other part. Then each side of the bend will move outwards, and thus the ring will tend towards a circular form again, though it possibly oscillates around it.

It is evident that whatever changes a hollow vortex may undergo the inner fluid can never mix with the outer, but the vortex surface forms a box through which not a drop of the fluid can escape however the surface may be twisted.

Thomson and others have pronounced all such discontinuous fluid motion to be unstable, and to prove this he supposes a depression to be made in the surface, which he then supposes to increase indefinitely. That some surfaces are thus unstable no one can doubt. But I do not think that all are so. Besides, if we have a vortex surface drifting around in a fluid, the latter having only an irrotational motion, the depression which will form in the surface at any time will have to satisfy certain conditions which may neutralize the previous effect. I have given my reasons for regarding the hollow vortex of uniform strength of surface as the form toward which hollow vortices tend when distorted. But I am not yet prepared to give further results, but will wait to obtain an exact solution of the problem. It is very probable that, when disturbed, oscillations are set up. All the principles necessary for the solution are developed in this paper.

If it is found that the hollow vortex is unstable, it may still be possible to build up a solid stable vortex on the principles here set forth. The condition thus developed for a vortex surface which shall not change its form by the mutual action of its parts is that the strength of the vortex surface shall be constant at every part and the surface a stream surface.

But in order that such a surface may exist, it must be remembered that the components of the vortex motion must satisfy the equation of continuity and the motion of the fluid inside the surface must be acyclic. This last condition is only necessary in order to avoid vortex motion in other parts than the sheet, which, however, may exist if we please. Having constructed such a sheet and calculated the fluid motion from it, it will have a potential everywhere without the sheet satisfying Laplace's equation, without vortex motion. If we construct a hollow vortex ring in this manner, with the condition that the surface be a stream surface, the fluid within the hollow part will have an acyclic motion, but such as not to affect the form of the surface. Thus the motion might be one of translation, and it would be thus possible to have a stable hollow vortex. The only form of surface which can exist without changing its form is thus symmetrical around the axis of motion, and has a cross-section which has not yet been investigated.

In certain cases the changes through which a hollow vortex goes are periodical, and it is a question whether, with a proper motion of translation, they are not always of this nature.

Let there be a distribution $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$ inside a given surface. Then F_n, G_n, H_n will have a potential satisfying Laplace's equation at all points where $\bar{F}_n = 0$, $\bar{G}_n = 0$, and $\bar{H}_n = 0$, of the value $\xi_{n,0} = -\chi_n$.

$$\xi_{n,0} = \frac{1}{4\pi} \iiint \left\{ \bar{F}_n \frac{d\frac{1}{r}}{dx} + \bar{G}_n \frac{d\frac{1}{r}}{dy} + \bar{H}_n \frac{d\frac{1}{r}}{dz} \right\} dxdydz,$$

or integrating by parts

$$\xi_{n,0} = \frac{1}{4\pi} \iint \frac{1}{r} \{ \bar{F}_n l + \bar{G}_n m + \bar{H}_n n \} dS - \frac{1}{4\pi} \iiint \frac{1}{r} \left\{ \frac{d\bar{F}_n}{dx} + \frac{d\bar{G}_n}{dy} + \frac{d\bar{H}_n}{dz} \right\} dxdydz.$$

If within the surface

$$\bar{F}_n = \frac{d\phi_n}{dx}, \qquad \bar{G}_n = \frac{d\phi_n}{dy}, \qquad \text{and } \bar{H}_n = \frac{d\phi_n}{dz}$$

we have

$$\xi_{n,0} = \frac{1}{4\pi} \iint \frac{\phi_n}{r^2} \cos \theta dS + \phi_n,$$

where θ is the angle between the normal to the surface and r, and the last term

disappears outside the surface where $\bar{F}_n$, $\bar{G}_n$, and $\bar{H}_n$ do not exist. Whence outside the surface

$$\xi'_{n,0} = \frac{1}{4\pi}\iint \frac{\phi_n}{r^2} \cos\theta dS$$

and inside

$$\xi' = \frac{1}{4\pi}\iint \frac{\phi_n}{r^2} \cos\theta dS + \phi_n.$$

The integral is also evidently different outside and inside the surface. We evidently have also

$$\Delta^2 \xi'_{n,0} = 0 \text{ outside the surface,}$$

and

$$\Delta^2 \xi_{n,0} = 0 \text{ inside the surface.}$$

Hence this is a case similar to the one before it, and the surface has a distribution over it of motion of the $(n+1)^{\text{th}}$ order given by the equations found there; this motion is confined to the surface, and forms closed circuits on it in order to satisfy the equation of continuity.

When motion of the $(n+1)^{\text{th}}$ order is so distributed over the surface that the component of the n^{th} order of motion is zero in the direction of the normal to the surface, then, from what we have before proved, the motions of the $(n+1)^{\text{th}}$ order and higher orders will be confined to the surface, and will not appear in the remainder of the fluid. Thus in the motion resulting from a solid body moving in the fluid, the motion of the zero order is zero at the surface, and so there is no motion of the first or higher orders throughout the fluid, and the motion of the zero order has a potential.

If the discontinuous system $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$, whose component is $\bar{M}_n$, are distributed over a surface with the resultant normal to the surface, $d\nu$ being the thickness of the surface, and $\bar{M}_n d\nu$ being constant over the surface, then we have seen that the shell so formed is equivalent to a line of $(n+1)^{\text{th}}$ motion around the edge.

If dS is the cross section of the line, we must have *

$$\bar{M}_n d\nu = M_{n+1} dS.$$

The linear distribution of M_{n+1} is then equivalent to the surface distribution of $\bar{M}_n$. There will be a potential of the n^{th} order throughout space, satisfying Laplace's equation: the motion of the $(n+1)^{\text{th}}$ and higher orders are confined to the boundary line of the surface, and are zero throughout the rest of the space.

* This is proved in the portion of this paper relating to the energy of the fluid.

So that all space except the line surely contains no motion of the n^{th} and lower orders, the n^{th} only having a potential which satisfies Laplace's equation but is cyclic.

The components of the $(n+1)^{\text{th}}$ order are readily obtained by differentiation of the equations

$$F_{n-1} = \frac{M_{n+1}dS}{4\pi} \int \frac{l}{r}\, ds,$$

$$G_{n-1} = \frac{M_{n+1}dS}{4\pi} \int \frac{m}{r}\, ds,$$

$$H_{n-1} = \frac{M_{n+1}dS}{4\pi} \int \frac{n}{r}\, ds,$$

where l, m, and n are evidently the direction cosines of the element of the line ds.

The scalar potential of the n^{th} order is simply proportional to the solid angle subtended by the line at the point, and can be computed by known methods. The components of the n^{th} order can also be simply obtained from the above components of the $(n-1)^{\text{th}}$ order by the proper differentiation.

As the motions of the $(n+1)^{\text{th}}$ order and higher are confined to the line, the total motion of the fluid can be represented not only by a line integral of the $(n+1)^{\text{th}}$ order, but also by a line integral of *any* of the higher orders. The exact form I have not yet investigated.

As an illustration of the methods here given, let us suppose that throughout a sphere of radius R there is a distribution of motion represented by $\bar{F}_n$, while $\bar{G}_n$ and $\bar{H}_n$ are zero. We shall then have

$$O_{n,2} = -\frac{\bar{F}_n}{24} \int\int\int r^3 dxdydz,$$

$$P_{n,2} = 0,$$

$$Q_{n,2} = 0,$$

where the integral is to be taken throughout the interior of the sphere. We thus find

Inside the Sphere. *Outside the Sphere.*

$$O_{n,2} = \frac{\pi \bar{F}_n}{1260}\{35R^6 + 105R^4r^2 + 21R^2r^4 - r^6\};\quad O'_{n,2} = \frac{\pi \bar{F}_n R^3}{630\, r}\{3R^4 + 42R^2r^2 + 35r^4\};$$

$$P_{n,2} = Q_{n,2} = P'_{n,2} = Q'_{n,2} = 0.$$

Since we have $\Delta^2 r^n = n(n+1)r^{n-2}$, we can write all the following quantities from differentiating these: —

$$O_{n,1} = \frac{\pi \bar{F}_n}{30}\{15R^4 + 10R^2r^2 - r^4\}; \qquad O'_{n,1} = \frac{2\pi R^3 \bar{F}_n}{15r}\{R^2 + 5r^2\};$$

$$P_{n,1} = Q_{n,1} = P'_{n,1} = Q'_{n,1} = 0;$$

$$O_{n,0} = \frac{2\pi \bar{F}_n}{3}\{3R^2 - r^2\}; \qquad O'_{n,0} = \frac{4}{3}\frac{\pi R^3 F_n}{r};$$

$$P_{n,0} = Q_{n,0} = P'_{n,0} = Q_{n,0} = 0;$$

$$\Delta^2 O_{n,(-1)} = 4\pi \bar{F}_n; \qquad \Delta^2 O'_{n,(-1)} = 0.$$

From these we have

$$\xi_{n,2} = \frac{\bar{F}_n r}{840}\{35R^4 + 14R^2r^2 - r^4\}\cos\theta; \qquad \xi'_{n,2} = \frac{\bar{F}_n R^3}{120}\{2R^2 + 5r^2\}\cos\theta;$$

$$\xi_{n,1} = \frac{\bar{F}_n r}{30}\{5R^2 - r^2\}\cos\theta; \qquad \xi'_{n,1} = \frac{\bar{F}_n R^3}{30r^2}\{5r^2 - R^2\}\cos\theta;$$

$$\xi_{n,0} = -\frac{\bar{F}_n}{3} r\cos\theta; \qquad \xi'_{n,0} = -\frac{\bar{F}_n R^3}{3}\frac{\cos\theta}{r^2}.$$

Whence we can write the whole system of vectors as follows: —

$$F_{n-4} = \frac{\bar{F}_n}{120}\{15R^4 + 10R^2r^2 - r^4\} - \frac{d\xi_{n,2}}{dx}; \quad F'_{n-4} = \frac{\bar{F}_n R^3}{30r}\{R^2 + 5r^2\} - \frac{d\xi'_{n,2}}{dx};$$

$$G_{n-4} = -\frac{d\xi_{n,2}}{dy}; \qquad G'_{n-4} = -\frac{d\xi'_{n,2}}{dy};$$

$$H_{n-4} = -\frac{d\xi_{n,2}}{dz}; \qquad H'_{n-4} = -\frac{d\xi'_{n,2}}{dz}.$$

$$F_{n-3} = 0; \qquad F'_{n-3} = 0;$$

$$G_{n-3} = -\frac{\bar{F}_n}{30}\{5R^2 - r^2\}z; \qquad G'_{n-3} = -\frac{\bar{F}_n R^3}{30r}\left\{\frac{R^2}{r^2} + 5\right\}z;$$

$$H_{n-3} = \frac{\bar{F}_n}{30}\{5R^2 - r^2\}y; \qquad H'_{n-3} = \frac{\bar{F}_n R^3}{30r}\left\{\frac{R^2}{r^2} + 5\right\}y.$$

$$F_{n-2} = \frac{\bar{F}_n}{6}\{3R^2 - r^2\} - \frac{d\xi_{n,1}}{dx}; \qquad F'_{n-2} = \frac{\bar{F}_n R^3}{3r} - \frac{d\xi'_{n,1}}{dx};$$

$$G_{n-2} = -\frac{d\xi_{n,1}}{dy}; \qquad G'_{n-2} = -\frac{d\xi'_{n,1}}{dy};$$

$$H_{n-2} = -\frac{d\xi_{n,1}}{dz}; \qquad H'_{n-2} = -\frac{d\xi'_{n,1}}{dz}.$$

$$F_{n-1} = 0; \qquad F'_{n-1} = 0;$$

$$G_{n-1} = \frac{\bar{F}_n}{3} z; \qquad G'_{n-1} = \frac{\bar{F}_n R^3}{3} \frac{z}{r^3};$$

$$H_{n-1} = -\frac{\bar{F}_n}{3} y; \qquad H'_{n-1} = -\frac{\bar{F}_n R^3}{3} \frac{y}{r^3}.$$

$$F_n = \bar{F}_n - \frac{d\xi_{n,0}}{dx} = \frac{2}{3} \bar{F}_n; \qquad F'_n = -\frac{d\xi_{n,0}}{dx} = \frac{\bar{F}_n R^3}{3} \frac{r^2 - 3x^2}{r^5};$$

$$G_n = -\frac{d\xi_{n,0}}{dy} = 0; \qquad G'_n = -\frac{d\xi_{n,0}}{dy} = -\frac{\bar{F}_n R^3}{3} \frac{3xy}{r^5};$$

$$H_n = -\frac{d\xi_{n,0}}{dz} = 0; \qquad H'_n = -\frac{d\xi_{n,0}}{dz} = -\frac{F_n R^3}{3} \frac{3xz}{r^5}.$$

The orders higher than these are only distributed over the surface, the next higher order being

$$F_{n+1}\, d\nu = 0,$$

$$G_{n+1}\, d\nu = -\frac{1}{3} \bar{F}_n \frac{z}{R},$$

$$H_{n+1}\, d\nu = \frac{1}{3} \bar{F}_n \frac{y}{R}.$$

The next higher order is distributed, as it were, on each side of the surface whose thickness is $d\nu$.

By giving to n different values up to $n = 4$, we can get various distributions of fluid motion, all of which satisfy the equation of continuity and are thus possible forms of motion. When $n = 4$, the motion of the fluid can be considered as due to a uniform distribution of motion of the fourth order, $\bar{F}_4$, throughout the sphere, the motions of the lower orders being distributed as given by the equations. Or the motion can be considered as due to a distribution of F_{n+1}, G_{n+1}, H_{n+1} over the surface of the sphere. When $n = 0$, the case is that of a sphere of liquid proceeding forward uniformly in the direction of the axis of x.

If we replace the sphere by a solid sphere, the motion of the fluid outside will remain the same as before.

The changes which a sphere moving in the manner described will undergo can be calculated from the drifting of the vortex sheet backwards, as shown further on.

The sphere will thus tend to flatten in the direction of the axis of x and broaden out in the other direction so as to form a figure somewhat similar to a prolate ellipsoid.

It is to be noted that the whole system of motions which I have given above have only required *one* integration, and in the general case would only require *three* for the complete determination of all the vectors up to the $2s$ order.

As another illustration, take a distribution of $\bar{F}_n$ along the axis of x, and make $s = 2$, as before. Let the area of the section of the small tube along the axis be a, and in the integration reject the part which becomes infinite. We thus find

$$O_{n,2} = \frac{\bar{F}_n a}{64} q^4 \log q.$$

where

$$q = \sqrt{y^2 + z^2},$$

whence

$$\xi_{n,2} = 0,$$

and

$$F_{n-4} = \frac{1}{4\pi} \Delta^2 O_{n,2} = \frac{\bar{F}_n a}{8} (q^2 + 2q^2 \log q),$$

$$G_{n-4} = 0,$$

$$H_{n-4} = 0;$$

$$F_{n-3} = 0,$$

$$G_{n-3} = \frac{\bar{F}_n a}{2} (1 + \log q) z,$$

$$H_{n-3} = -\frac{\bar{F}_n a}{2} (1 + \log q) y;$$

$$F_{n-2} = -\frac{\bar{F}_n a}{2} \{3 + 2 \log q\},$$

$$G_{n-2} = 0,$$

$$H_{n-2} = 0;$$

$$F_{n-1} = 0,$$

$$G_{n-1} = -\bar{F}_n a \frac{z}{q^2},$$

$$H_{n-1} = +\bar{F}_n a \frac{y}{q^2}.$$

The calculation of F_n, G_n, H_n makes them zero for all points of space except the axis of x, just as it should.

But we can also consider them as distributed over any surface enclosing the axis of x.

Taking a stream surface for F_{n-1}, G_{n-1}, H_{n-1} which will be any surface of revolution around the axis, the proper distribution of F_n, G_n, H_n over the surface will cause the above values outside the surface, and a zero value of F_{n-1}, G_{n-1}, H_{n-1} inside.

Thus, take a circular cylinder of radius b. We must distribute $\bar{F}_n a$ over the surface of thickness $d\nu$. Therefore we have

$$2\pi b \bar{F}'_n d\nu = \bar{F}_n a,$$

$$\bar{F}'_n d\nu = \frac{\bar{F}_n a}{2\pi b}.$$

Beside this distribution of $\bar{F}_n$ we can distribute it uniformly in the interior of the circular cylinder, or in any other way suggested by the equations. In the case of uniform distribution throughout the cylinder we can replace it by a surface distribution of F_{n+1}, G_{n+1}, H_{n+1} and so on ad infinitum, the exterior distribution of velocities being the same, but the interior being different.

The Action of Forces on Fluids.

Let a system of forces whose components at the point x, y, z are $\bar{X}$, $\bar{Y}$, $\bar{Z}$ act on a fluid, and let p be the pressure of the fluid at the same point. The hydrodynamical equations of Euler are then,

$$\frac{1}{\rho}\frac{dp}{dx} = \bar{X}_0 - \frac{dF_0}{dt} - F_0\frac{dF_0}{dx} - G_0\frac{dF_0}{dy} - H_0\frac{dF_0}{dz},$$

$$\frac{1}{\rho}\frac{dp}{dy} = \bar{Y}_0 - \frac{dG_0}{dt} - F_0\frac{dG_0}{dx} - G_0\frac{dG_0}{dy} - H_0\frac{dG_0}{dz},$$

$$\frac{1}{\rho}\frac{dp}{dz} = \bar{Z}_0 - \frac{dH_0}{dt} - F_0\frac{dH_0}{dx} - G_0\frac{dH_0}{dy} - H_0\frac{dH_0}{dz}.$$

It is usual in treatises on hydrodynamics to consider cases where $\bar{X}_0$, $\bar{Y}_0$, $\bar{Z}_0$ have a potential, and it is there stated that vortex motion cannot be produced by such forces. But if we consider that forces which have a scalar potential are

such as are produced by direct attraction to or repulsion from points distributed throughout the fluid, we see that conservative forces acting in an unlimited medium can never produce *any motion whatever*, but only influence the pressure.

Thus the equations as they stand contain much that is superfluous, and the motion will be the same in every respect if we differentiate in such a manner as to eliminate all portions of X, Y, Z which depend on a scalar potential.

Let us write

$$W = \frac{p}{S} + \tfrac{1}{2} M_0^2 ;$$

then the equations can be put in the form*

$$\frac{dW}{dx} - \overline{X}_0 + \frac{dF_0}{dt} = G_0 H_1 - H_0 G_1,$$

$$\frac{dW}{dy} - \overline{Y}_0 + \frac{dG_0}{dt} = H_0 F_1 - F_0 H_1,$$

$$\frac{dW}{dz} - \overline{Z}_0 + \frac{dH_0}{dt} = F_0 G_1 - G_0 F_1.$$

Let us now write

$$X_1 = \frac{d\overline{Z}_0}{dy} - \frac{d\overline{Y}_0}{dz},$$

$$Y_1 = \frac{d\overline{X}_0}{dz} - \frac{d\overline{Z}_0}{dx},$$

$$Z_1 = \frac{d\overline{Y}_0}{dx} - \frac{d\overline{X}_0}{dy},$$

and also

$$X_2 = \frac{dZ_1}{dy} - \frac{dY_1}{dz} = -\Delta^2 \overline{X}_0 + \frac{d}{dx}\left(\frac{d\overline{X}_0}{dx} + \frac{d\overline{Y}_0}{dy} + \frac{d\overline{Z}_0}{dz}\right),$$

$$Y_2 = \frac{dX_1}{dz} - \frac{dZ_1}{dx} = -\Delta^2 \overline{Y}_0 + \frac{d}{dy}\left(\frac{d\overline{X}_0}{dx} + \frac{d\overline{Y}_0}{dy} + \frac{d\overline{Z}_0}{dz}\right),$$

$$Z_2 = \frac{dY_1}{dx} - \frac{dX_1}{dy} = -\Delta^2 \overline{Z}_0 + \frac{d}{dz}\left(\frac{d\overline{X}_0}{dx} + \frac{d\overline{Y}_0}{dy} + \frac{d\overline{Z}_0}{dz}\right),$$

etc. etc.

Now I have shown in the theory of vector quantities that every system of discontinuous vectors can be replaced by another system which satisfies the equation of continuity, provided we can show a physical reason for the vectors

* Given in a more restricted form by Lamb, in his "Treatise on the Motion of Fluids," p. 241.

satisfying that equation. In the case under consideration, the pressure conducts the force applied at one point to another, and the whole system of forces so applied to the fluid must satisfy the equation of continuity. Hence for the system $\bar{X}_n$, $\bar{Y}_n$, $\bar{Z}_n$, can be substituted the system

$$X_n = \bar{X}_n + \frac{d\chi_n}{dx},$$

$$Y_n = \bar{Y}_n + \frac{d\chi_n}{dy},$$

$$Z_n = \bar{Z}_n + \frac{d\chi_n}{dz},$$

where

$$\chi_n = -\frac{1}{4\pi}\iiint\frac{1}{r}\left(\frac{d\bar{X}_n}{dx} + \frac{d\bar{Y}_n}{dy} + \frac{d\bar{Z}_n}{dz}\right) dxdydz.$$

When this substitution for the zero order is made in the original equations, the only portion of the pressure that will remain will be that which arises from the motion of the fluid and not from the applied forces. In this case we can simply write

$$X_2 = -\Delta^2 X_0;\quad Y_2 = -\Delta^2 Y_0;\quad Z_2 = -\Delta^2 Z_0.$$

Differentiating the third of our equations with respect to y and the second with respect to z, and subtracting, we can write the first of the following series of equations; the other two can be written from symmetry: —

$$X_1 = \frac{\delta F_1}{\delta t} - F_1\frac{dF_0}{dx} - G_1\frac{dF_0}{dy} - H_1\frac{dF_0}{dz},$$

$$Y_1 = \frac{\delta G_1}{\delta t} - F_1\frac{dG_0}{dx} - G_1\frac{dG_0}{dy} - H_1\frac{dG_0}{dz},$$

$$Z_1 = \frac{\delta H_1}{\delta t} - F_1\frac{dH_0}{dx} - G_1\frac{dH_0}{dy} - H_1\frac{dH_0}{dz},$$

where the symbol δ refers to the *moving* element and has the well-known value

$$\frac{\delta}{\delta t} = \frac{d}{dt} + F_0\frac{d}{dx} + G_0\frac{d}{dy} + H_0\frac{d}{dz}.$$

Performing the same operation on these, we have the equations

$$X_2 = \frac{\delta F_2}{\delta t} - F_2 \frac{dF_0}{dx} - G_2 \frac{dF_0}{dy} - H_2 \frac{dF_0}{dz} + 2 \left\{ \begin{array}{l} \dfrac{dF_0}{dy}\dfrac{dF_1}{dz} - \dfrac{dF_1}{dy}\dfrac{dF_0}{dz} \\ + \dfrac{dG_0}{dy}\dfrac{dG_1}{dz} - \dfrac{dG_1}{dy}\dfrac{dG_0}{dz} \\ + \dfrac{dH_0}{dy}\dfrac{dH_1}{dz} - \dfrac{dH_1}{dy}\dfrac{dH_0}{dz} \end{array} \right\},$$

$$Y_2 = \frac{\delta G_2}{\delta t} - F_2 \frac{dG_0}{dx} - G_2 \frac{dG_0}{dy} - H_2 \frac{dG_0}{dz} + 2 \left\{ \begin{array}{l} \dfrac{dF_0}{dz}\dfrac{dF_1}{dx} - \dfrac{dF_1}{dz}\dfrac{dF_0}{dx} \\ + \dfrac{dG_0}{dz}\dfrac{dG_1}{dx} - \dfrac{dG_1}{dz}\dfrac{dG_0}{dx} \\ + \dfrac{dH_0}{dz}\dfrac{dH_1}{dx} - \dfrac{dH_1}{dz}\dfrac{dH_0}{dx} \end{array} \right\},$$

$$Z_2 = \frac{\delta H}{\delta t} - F_2 \frac{dH_0}{dx} - G_2 \frac{dH_0}{dy} - H_2 \frac{dH_0}{dz} + 2 \left\{ \begin{array}{l} \dfrac{dF_0}{dx}\dfrac{dF_1}{dy} - \dfrac{dF_1}{dx}\dfrac{dF_0}{dy} \\ + \dfrac{dG_0}{dx}\dfrac{dG_1}{dy} - \dfrac{dG_1}{dx}\dfrac{dG_0}{dy} \\ + \dfrac{dH_0}{dx}\dfrac{dH_1}{dy} - \dfrac{dH_1}{dx}\dfrac{dH_0}{dy} \end{array} \right\}.$$

When a fluid in motion is left to itself, the problem of the changes which it undergoes has never been satisfactorily solved, even in the case of a single vortex ring left in space by itself, and the principles to guide one in the solution of the problem have not been very satisfactorily given.

When the forces acting on the fluid are zero, we have

$$\frac{\delta F_1}{\delta t} = F_1 \frac{dF_0}{dx} + G_1 \frac{dF_0}{dy} + H_1 \frac{dF_0}{dz},$$

$$\frac{\delta G_1}{\delta t} = F_1 \frac{dG_0}{dx} + G_1 \frac{dG_0}{dy} + H_1 \frac{dG_0}{dz},$$

$$\frac{\delta H_1}{\delta t} = F_1 \frac{dH_n}{dx} + G_1 \frac{dH_0}{dy} + H_1 \frac{dH_0}{dz}.$$

These equations contain the whole dynamics of the subject, and simply show that the product of the strength of the first order of motion by its cross section is constant. This principle then, applied in the proper way, contains the whole of the dynamics of a perfect fluid.

The process indicated by the equations is as follows: Having given certain values of the components of the velocities, F_0, G_0, H_0, which satisfy the equation of continuity, we calculate from these the distribution of the first order of motion.

The variation of this first order of motion in each element as it drifts along is given by the above equations, or the variation of motion in any fixed element is given by these modified equations.

The variation of the fluid motion can then be calculated from these. Again applying the method, we could get a still further change, and if we continued this step by step process indefinitely we might trace out the whole fluid motion. But we might originally obtain F_0, G_0, H_0 as functions of x, y, z, and t, so that for $t =$ constant, they should satisfy the equation of continuity, and for t variable, the above equations.

The result so found would give all the changes which a given system would undergo which was started at a given time in any one of the configurations. *But it is to be particularly noted that in using these equations we must always descend to such small elements of the fluid that discontinuity is avoided.*

Thus we should never treat a single vortex filament by itself, but should descend to the still smaller filaments of which the vortex filament is composed, and apply our equations to these.

In this manner the problem of the motion of a single vortex filament becomes perfectly determinate instead of indeterminate as before. And I believe that a recognition of this subject will lead to extremely important results in this subject. It is, then, in the study of these differential equations that the final solution is to be obtained.

We can always tell from the equations in which direction the system tends to change, and can thus base a theory of the stability of fluid motion on them. But I have not yet attempted to find any solutions. It is to be noted that the equations are satisfied at all points of space for which the first order of motion is zero, and so we can always confine our attention to points where it exists.

But we are able to regard the matter from another point of view. When no external forces act on the fluid, the equations of Euler become

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{\delta F_0}{\delta t},$$

$$\frac{1}{\rho}\frac{dp}{dy} = -\frac{\delta G_0}{dt},$$

$$\frac{1}{\rho}\frac{dp}{dz} = -\frac{\delta H_0}{dt},$$

where

$$-\frac{\delta F_0}{\delta t}, \qquad -\frac{\delta G_0}{\delta t}, \qquad -\frac{\delta H_0}{\delta t},$$

are the forces of acceleration of the fluid. Hence we have

$$\frac{d}{dy}\frac{\delta H_0}{\delta t} - \frac{d}{dz}\frac{\delta G_0}{\delta t} = 0,$$

$$\frac{d}{dz}\frac{\delta F_0}{\delta t} - \frac{d}{dx}\frac{\delta H_0}{\delta t} = 0,$$

$$\frac{d}{dx}\frac{\delta G_0}{\delta t} - \frac{d}{dy}\frac{\delta F_0}{\delta t} = 0.$$

These equations simply express the fact that in a fluid not acted upon by external forces the forces of acceleration are acyclic. One of the most interesting cases is that of a surface of discontinuity. In this case the equations assume the form

$$\left(\frac{\delta H_0'}{\delta t} - \frac{\delta H_0}{\delta t}\right)\frac{d\nu}{dy} - \left(\frac{\delta G_0'}{\delta t} - \frac{\delta G_0}{\delta t}\right)\frac{d\nu}{dz} = 0,$$

$$\left(\frac{\delta F_0'}{\delta t} - \frac{\delta F_0}{\delta t}\right)\frac{d\nu}{dz} - \left(\frac{\delta H_0'}{\delta t} - \frac{\delta H_0}{\delta t}\right)\frac{d\nu}{dx} = 0,$$

$$\left(\frac{\delta G_0'}{\delta t} - \frac{\delta G_0}{\delta t}\right)\frac{d\nu}{dx} - \left(\frac{\delta F_0'}{\delta t} - \frac{\delta F_0}{\delta t}\right)\frac{d\nu}{dy} = 0.$$

To get the meaning of these equations let us transform them until the normal to the surface is in the direction of the axis of X. We then have

$$\frac{\delta H_0'}{\delta t} - \frac{\delta H_0}{\delta t} = 0,$$

$$\frac{\delta G_0'}{\delta t} - \frac{\delta G_0}{\delta t} = 0.$$

And we also have for the continuity of the fluid

$$F_0' - F_0 = 0.$$

At all points of the fluid where there is no discontinuity, but the fluid velocities are obtained from a potential, these equations are satisfied. It is only where vortices exist that there is a possibility of the equations not holding.

The equations show that there must be no discontinuity in the forces of acceleration at the surface and parallel to it, though there may be in the direction normal to the surface.

If we then have a surface of discontinuity of this nature, we must then add to the forces of acceleration in the interior of the surface others so as to make the system continuous at the surface in all directions except that perpendicular to the surface. This is the same thing as saying that forces must exist throughout the interior of the surface tending to change the configuration.

These forces must evidently be acyclic within the surface, and have the proper value at the surface, and so are perfectly determined.

As an illustration of these methods of finding what dynamical changes will take place in a fluid, let us take the case of a single vortex filament along the axis of X. The fluid velocity will be inversely as the distance from the axis, or

$$F_0 = 0,$$
$$G_0 = F_1 a \frac{z}{q^2},$$
$$H_0 = -F_1 a \frac{y}{q^2},$$

where $q = \sqrt{y^2 + z^2}$, and a is the sectional area of the filament.

Let us take a stream surface bounded by $x = b$ and $x = 0$ and $q = c$ and $q = e$, and distribute $F_1 a$ over it according to our equations. Then, on taking away the original distribution of vortex motion, the fluid will move within the surface the same as before, but without will be at rest. We immediately see that the system is not in equilibrium, for the centrifugal force of the moving liquid remains unbalanced by the fluid pressure. We readily see that the whole ring will expand indefinitely.

The second method expresses the fact as follows: For the surface $q = c$ and $q = e$ the vortex strength has of itself no tendency to vary.

But on the plane surfaces we have

$$\frac{\delta F_1}{\delta t} = 0,$$
$$\frac{\delta G_1}{\delta t} = -F_1 a \left\{ G_1 \frac{2yz}{q^4} + H_1 \frac{z^2 - y^2}{q^4} \right\},$$
$$\frac{\delta H_1}{\delta t} = -F_1 a \left\{ G_1 \frac{z^2 - y^2}{q^4} - H_1 \frac{2yz}{q^4} \right\}.$$

Now the distribution of G_1 and H_1 is in the radial direction, and so these equations show that the direction of the vortex elements tends to change to one in a direction perpendicular to q. Thus we arrive at the same direction of change as before.

By the other method of looking upon the problem, we see that forces of acceleration must exist in the stationary fluid, and so the stationary portion will tend to move. The direction is readily determined.

We are now prepared to state what the conditions are that there shall be no tendency to change from its present configuration.

For a surface of discontinuity the condition is simply that the surface be a stream surface and that the strength of the surface, $Md\nu$, be constant, or that the fluid velocity on the two sides of the sheet be everywhere equal and opposite in direction.

This investigation has led us to look upon the subject of vortex motion from a broader point of view than before. For we have seen that if we reckon up the amount of this motion by the number of fluid elements multiplied by the strength of the motion in them, then the motion so reckoned is being constantly created and destroyed by the other fluid motions. But if we reckon the quantity of vortex motion by its surface integral taken across its cross section, or by its *circulation,* then the statement that it is indestructible by any motion lower than itself is perfectly çorrect. And this latter definition of the *quantity* of any vector is so important in the theory of the replacement of one system of vectors by another, that I propose that the term obtain general use.

The equilibrium of fluid motion can evidently be either stable, unstable, or neutral. Thus an infinitely long cylinder of fluid revolving around the axis in a medium at rest would evidently be in unstable equilibrium. Thomson has given the criteria for such cases in terms of the energy of the system, and I am not yet prepared to discuss the subject further.

But his conclusion as to the instability of all cases of discontinuous fluid motion I am not willing to admit. The case of the hollow vortex, it seems to me, shows that there can be stable forms. For the changes which the system undergoes are toward stability, seeing that the vortex distribution of the sheet tends to become uniform, which is a case of equilibrium.

Again, the sheet can never be broken and the inside fluid mix with the outside. Hence I am of the opinion that a hollow vortex ring is stable, and always tends to the form where the vortex strength of the sheet is uniform over the surface. But the subject should be carefully examined before a final decision can be reached.

To determine the pressure in fluid motion we can use the equations

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{\delta F_0}{\delta t},$$

$$\frac{1}{\rho}\frac{dp}{dy} = -\frac{\delta G_0}{\delta t},$$

$$\frac{1}{\rho}\frac{dp}{dz} = -\frac{\delta H_0}{\delta t},$$

whence we have

$$\Delta^2 p = -\rho\left\{\frac{d}{dx}\frac{\delta F_0}{\delta t} + \frac{d}{dy}\frac{\delta G_0}{\delta t} + \frac{d}{dz}\frac{\delta H_0}{\delta t}\right\},$$

whence

$$v = \frac{\rho}{4\pi}\iiint\frac{1}{r}\left\{\frac{d}{dx}\frac{\delta F_0}{\delta t} + \frac{d}{dy}\frac{\delta G_0}{\delta t} + \frac{d}{dz}\frac{\delta H_0}{\delta t}\right\}dxdydz.$$

This can be put in the form

$$p = -\frac{\rho}{2\pi}\iiint\left\{\frac{dF_0}{dy}\frac{dG_0}{dx} - \frac{dF_0}{dx}\frac{dG_0}{dy} + \frac{dF_0}{dz}\frac{dH_0}{dx} - \frac{dF_0}{dx}\frac{dH_0}{dz} + \frac{dG_0}{dz}\frac{dH_0}{dy} - \frac{dG_0}{dy}\frac{dH_0}{dz}\right\}dxdydz.$$

We may also write a very interesting form from integration by parts

$$p = -\frac{\rho}{4\pi}\iiint\left\{\frac{\delta F_0}{\delta t}\frac{d\frac{1}{r}}{dx} + \frac{\delta G_0}{\delta t}\frac{d\frac{1}{r}}{dy} + \frac{\delta H_0}{\delta t}\frac{d\frac{1}{r}}{dz}\right\}dxdydz.$$

If the forces of acceleration have a potential throughout a given region, these reduce to surface integrals. If the forces of acceleration form closed circuits, no fluid pressure exists; that is, if the forces of acceleration satisfy the equation of continuity.

In this case, since we have

$$\frac{d}{dx}\frac{dF_0}{dt} + \frac{d}{dy}\frac{dG_0}{dt} + \frac{d}{dz}\frac{dH_0}{dt} = 0,$$

if we write

$$R_0 = F_0\frac{dF_0}{dx} + G_0\frac{dF_0}{dy} + H\frac{dF_0}{dz},$$

$$S_0 = F_0\frac{dG_0}{dx} + G_0\frac{dG_0}{dy} + H_0\frac{dG_0}{dz},$$

$$T_0 = F_0\frac{dH_0}{dx} + G_0\frac{dH_0}{dy} + H_0\frac{dH_0}{az},$$

we shall then have

$$v = \frac{\rho}{4\pi}\iiint\frac{1}{r}\left\{\frac{dR}{dx} + \frac{dS}{dy} + \frac{dT}{dz}\right\}dxdydz,$$

whence

$$p = -\frac{\rho}{4\pi}\iiint\left\{L_0\frac{d\frac{1}{r}}{dx} + M_0\frac{d\frac{1}{r}}{dy} + N_0\frac{d\frac{1}{r}}{dz}\right\}dxdydz.$$

If M_1 is the resultant vortex motion and M_0 the resultant ordinary motion, and we draw ν in the direction of M_0, we can write the equation in the form

$$p = \frac{\rho}{2\pi}\iiint\frac{1}{r}\left\{\left(\frac{dM_0}{d\nu}\right)^2 - M_1^2\right\}dxdydz,$$

or, as we prefer to write it,

$$\Delta^2 p = 2\rho\left\{M_1^2 - \left(\frac{dM_0}{d\nu}\right)^2\right\}.$$

If points attracting as the square of the distance be distributed through space with a density

$$2\left\{\left(\frac{dM_0}{d\nu}\right)^2 - M_1^2\right\}$$

at every point, then the pressure would be the same as that due to the motion of the fluid.

In this sense the above expression may be considered as the *source* of the fluid pressure.

Let us now take the equations of page 260, and write

$$I = G_0 H_1 - H_0 G_1,$$

$$J = H_0 F_1 - F_0 H_1,$$

$$K = F_0 G_1 - G_0 F_1,$$

whence we have, calling C a constant,

$$p = C - \frac{\rho}{2} M_0^2 - \frac{\rho}{4\pi} \iiint \frac{1}{r} \left\{ \frac{dI}{dx} + \frac{dJ}{dy} + \frac{dK}{dz} \right\} dxdydz,$$

which can be put in the form

$$p = C - \frac{\rho}{2} M_0^2 - \frac{\rho}{4\pi} \iiint \frac{1}{r} \{ M_1^2 - F_0 F_2 - G \ G_2 - H_0 H_2 \} \, dxdydz.$$

We can also, from integration by parts, as the surface integral vanishes when the integral is taken throughout space, put

$$p = C - \frac{\rho}{2} M_0^2 + \frac{\rho}{4\pi} \iiint \left\{ I \frac{d\frac{1}{r}}{dx} + J \frac{d\frac{1}{r}}{dy} + K \frac{d\frac{1}{r}}{dz} \right\} dxdydz.$$

This expression can be written in the form

$$p = C - \frac{\rho}{2} M_0^2 + \phi,$$

where ϕ satisfies Laplace's equation at all points where no vortices exist. In the expression as ordinarily given, ϕ is the potential of the applied forces, but here it is given in terms of the vortex motion.

Some of these expressions for the pressure are similar to those obtained by Mr. Craig, and published in the Journal of the Franklin Institute.